n J Phys 80 (9), 937-941 (2006)

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PHYSICAL REVIEW VOLUME 98, NUMBER 2 JANUARY 15, 1953

Arbitrary Concentrations of Matter and the Schwarzschild Singularity

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(Received June 13, 1952)

A nonstatic solution of the Einstein gravitational equations representing a spherically symmetric cluster of radially moving particles in an otherwise empty space is obtained While it has been presumed by Einstein that the Schwarzschild singularity is physically unattainable as matter cannot be concentrated arbitrarily, the present solution seems to show that there is no theoretical limit to the degree of concentration, and that the Schwarzschild singularity has no physical reality as it occurs only in some particular coordinate systems. Incidentally, it is shown that in case of spherical symmetry the condition of conservation of gravitational energy of an isolated system of fluid material is equivalent to the vanishing of pressure at the boundary.

1. INTRODUCTION

THE Schwarzschild field for a mass particle,

$$ds^2 = -(1+m/2r)^4(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2) + \frac{(1-m/2r)^2dt^2}{(1+m/2r)^2}, \quad (1)$$

has singularities at $r=0$ and $r=m/2$. While it is not unnatural to identify the singularity at the origin of the spatial coordinate system with the mass particle, the "Schwarzschild singularity" at $r=m/2$ (corresponding to the vanishing of g_{44}) has been the subject of considerable speculation. Considering the field inside matter, Schwarzschild showed that if one considers an incompressible perfect fluid ($T_k{}^i = -p\delta_k{}^i$, for $i, k=1, 2, 3$, $T_4{}^4=\rho=$ const.), such a singularity corresponding to the vanishing of g_{44} can indeed be attained if the size of a sphere of given density be sufficiently large. However, as pointed out by Laue, the assumption of incompressibility is not consistent with the ideas of the theory of relativity. In order to avoid this difficulty, Einstein[1] has more recently examined the problem by considering a spherically symmetric assembly of particles moving in randomly oriented circles around a common center and in arbitrary phases. From the condition that the geodesics of the particles must be time-like, Einstein finds that there is a limit to the degree of concentration of matter, and it then follows that if matter be introduced in this particular form, the Schwarzschild singularity is physically unattainable. Further Einstein has expressed the view that it is not "subject to reasonable doubt that more general cases will have analogous results." However, the following considerations throw doubts on this presumption and have led to the present investigation.

2. FUNDAMENTAL IDEA OF THE PRESENT PAPER

If one considers the cosmologic solution corresponding to a spherically symmetric cluster of particles falling

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[1] A. Einstein, Ann. Math. 40, 922 (1939).

freely towards the center,[2] one finds that it is possible to reach any degree of concentration, there being no singularity except for infinite concentration. Now, suppose, we cut out a part R of such a contracting field and place it in an outside empty space Outside we expect to have the Schwarzschild field,[3] while inside we expect the cosmologic solution If the initial value of the radius of R be greater than that for Schwarzschild singularity (i.e , the concentration is less than the critical value required for this singularity), then there would be no singularity anywhere in the field. As the interior field goes on contracting, a singularity would appear in the exterior Schwarzschild solution as the radius crosses a certain critical value, however, so far as the interior field is concerned, no such singularity appears at any phase of its career Such a situation is obviously inconsistent, so that either it is impossible to fit such a contracting field with the outside empty space for arbitrary concentrations or else the Schwarzschild singularity is only a property of some particular coordinate systems and would not appear in other properly chosen coordinate systems

Einstein and Straus[4] have shown that in general it appears from the theory of differential equations that the cosmologic solution can be fitted to the Schwarzschild field; however, they have not considered whether the transformations which make the two solutions continuous are real and have also not considered the role of the Schwarzschild singularity in this problem of fit. In the present paper we shall show that it is possible to obtain a real solution of the gravitational equations for empty space which is continuous with the cosmologic solution at a time dependent boundary. No singularity corresponding to the Schwarzschild singularity appears at any phase in the exterior field for any arbitrary finite concentration in the cluster. The Schwarzschild singularity thus appears to be only a property of particular coordinate systems,[5] and there seems to be no theoretical limit to the degree of concentration

[1] In the usual cosmologic solution for an expanding universe, the "particles" are running away from the center However, the general theory of relativity is indifferent to the direction of time flux, and we are here considering the solution corresponding, so to say, to a contracting universe

[2] Any spherically symmetric solution in empty space is known to be reducible to the Schwarzschild field R Birkhoff, *Relativity and Modern Physics* (Harvard University Press, Cambridge, 1923), p 253

[4] A. Einstein and E G Straus, Revs Modern Phys 17, 120 (1945), 18, 148 (1946).

[5] Some earlier workers also have given transformations which become singular at the Schwarzschild singularity, whereby the Schwarzschild singularity may be abolished. G. Lemaitre, Ann. soc sci de Bruxelles, Ser. I 53, 51 (1933), J L. Synge, Proc. Roy. Irish Acad. A53, 84–114 (1950).

3. FIELD EQUATIONS AND THEIR INTEGRATION

With the isotropic spherically symmetric line element,

$$ds^2 = -e^{\mu}(dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)+e^{\nu}dt^2, \qquad (2)$$

$$\mu=\mu(r,t), \quad \nu=\nu(r,t),$$

the Einstein gravitational equations give[6]

$$-8\pi T_1{}^1 = e^{-\mu}[\mu'^2/4+\mu'\nu'/2+(\mu'+\nu')/r] - e^{-\nu}(\mu+\tfrac{3}{4}\dot{\mu}^2-\dot{\mu}\nu/2), \qquad (3)$$

$$-8\pi T_2{}^2 = -8\pi T_3{}^3 = e^{-\mu}\left(\frac{\mu''+\nu''}{2}+\frac{\nu'^2}{4}+\frac{\mu'+\nu'}{2r}\right) - e^{-\nu}\left(\ddot{\mu}+\tfrac{3}{4}\dot{\mu}^2-\frac{\mu\nu}{2}\right), \qquad (4)$$

$$8\pi T_4{}^4 = -e^{-\mu}(\mu''+\tfrac{1}{4}\mu'^2+2\mu'/r)+\tfrac{3}{4}e^{-\nu}\dot{\mu}^2, \qquad (5)$$

$$8\pi T_{14} = -(\dot{\mu}'-\tfrac{1}{2}\dot{\mu}\nu'), \qquad (6)$$

where dashes and dots denote differentiations with respect to r and t, respectively.

We now consider that in the region $0\leqslant r\leqslant r_1$, we have a spherically symmetric cluster of particles falling freely towards the origin Such a situation is obviously obtained if we assume[7] that in this region we have the well-known solution

$$\left.\begin{aligned} e^{\mu} &= e^{q}\Big/\left(1+\frac{zr^2}{4R^2}\right)^2 \\ e^{\nu} &= 1 \end{aligned}\right\} \quad \text{for} \quad r\leqslant r_1, \qquad (7)$$

where e^q is a decreasing function of t alone, $z=+1$, 0 or -1, and we take $r_1<2R$, so that so long as e^q is nonvanishing, there is no singularity in this region.

The coordinate system is co-moving, and the gravitational equations reduce to

$$0=-ze^{-q}/R^2-\ddot{q}-\tfrac{3}{4}\dot{q}^2, \qquad (8)$$

$$8\pi\rho = 3ze^{-q}/R^2+\tfrac{3}{4}\dot{q}^2, \qquad (9)$$

where $\rho=T_4{}^4$ is the density $=nm$, n being the number density of the particles and m the mass of each particle Equations (8) and (9) together give

$$\ddot{q}+\tfrac{1}{2}\dot{q}^2=-8\pi\rho/3, \qquad (10)$$

so that, ρ being essentially positive, $\ddot{q}$ is always negative Thus q cannot have any minimum[8], and the system once contracting ($\dot{q}$ negative) would go on contracting to infinite concentration

[6] R. C Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, London, 1934), p 252.

[7] With the form of solution assumed, ρ is a function of t alone A spatially nonuniform distribution of particles (retaining spherical symmetry) would, however, lead to the same results so far as our investigation is concerned.

[8] A. Einstein, *The Meaning of Relativity* (Methuen and Company, Ltd., London, 1950), fourth edition, p 113.

We next consider the field in the empty space $r \geqslant r_1$. The metric tensor must satisfy the field equations with $T_\nu^\mu = 0$ and along with the first derivatives must be continuous with (7) at $r = r_1$. Equation (6) gives on integration, $e^{-\nu}\dot{\mu}^2 = \varphi(t)$, where $\varphi(t)$ is a function of t alone. The condition of continuity now requires $\varphi(t) = \dot{q}^2$, so that we have

$$e^{-\nu}\dot{\mu}^2 = \dot{q}^2. \tag{11}$$

Using this result in Eq (5), we get

$$0 = -e^{-\mu}(\mu'' + \tfrac{1}{4}\mu'^2 + 2\mu'/r) + \tfrac{3}{4}\dot{q}^2.$$

Now, with the substitution[9]

$$e^{\mu}r^2 = \xi^4, \quad x = \log r, \tag{12}$$

the above equation becomes

$$\partial^2\xi/\partial x^2 = \tfrac{1}{4}\xi + \tfrac{3}{16}\dot{q}^2\xi^5; \tag{13}$$

and, on integration, we get

$$(\partial\xi/\partial x)^2 = \tfrac{1}{4}\xi^2 + \tfrac{1}{16}\dot{q}^2\xi^6 + K, \tag{14}$$

where K is apparently a function of t alone.

The corresponding equation in the region $0 \leqslant r \leqslant r_1$ would be

$$(\partial\xi/\partial x)^2 = \tfrac{1}{4}\xi^2 + \tfrac{1}{16}q^2\xi^6 - \tfrac{2}{3}\pi\rho\xi^6, \tag{15}$$

the function corresponding to K being here put equal to zero to avoid a singularity at the origin. Now the continuity of μ and μ' lead to the continuity of ξ^2 and $(\partial\xi/\partial x)^2$. Consequently, from (14) and (15),

$$K = -\left(\frac{2\pi\rho\xi^6}{3}\right)_{r=r_1} = -\frac{2\pi}{3}\rho\frac{e^{3q/2}}{(1+zr_1^2/4R^2)^3}r_1^2. \tag{16}$$

It is easy to see from (8) and (9) that $\rho e^{3q/2}$ is a constant independent of time as well, so that K also is a constant.

Equation (14) can be put into the Weierstrass form

$$(\partial y/\partial u)^2 = 4y^3 - q_2 y - q_3, \tag{17}$$

where

$$y = \frac{-K^{\frac{1}{2}}}{\xi^2} + \frac{1}{12K^{\frac{1}{2}}}, \qquad q_2 = \frac{1}{12K^{4/3}}, \tag{18}$$

$$q_3 = -(1/6^3K^3 + \dot{q}^2/4), \quad u = K^{\frac{1}{2}}\log r.$$

The solution can now be expressed in terms of Weierstrass's elliptic function,

$$y = \wp(u - u_1 + c, q_2, q_3), \tag{19}$$

where

$$u_1 = K^{\frac{1}{2}}\log r_1 \quad \text{and} \quad c = \wp^{-1}(y_1),$$

y_1 being the value of y at $r = r_1$ determined from (7), (12), and (18).

Thus, in view of Eqs. (11) and (12), the field in the empty space $r \geqslant r_1$ is given by

$$e^{\mu} = \xi^4/r^2, \quad e^{\nu} = (4\dot{\xi}/\dot{q}\xi)^2, \tag{20}$$

with ξ given by (18) and (19). It is easy to see that (20) satisfies all the field Eqs. (3)–(6) and the necessary boundary conditions. For, if one substitutes from (20) and (14) in (3), one finds that the necessary and sufficient condition that T_1^1 should vanish is that K must be independent of t. This condition, as we have already noted, is satisfied. With (3), (5), and (6) now satisfied, (4) is satisfied in view of the divergence identity.

The choice of c in (19) and the fixing of K by (16) have already ensured the continuity of μ and μ'. If therefore the clocks on the two sides of the boundary be synchronized,[10] μ and $\dot{\mu}'$ would also be continuous. Equations (11) and (6) then show that ν and ν' are also continuous.

4. TRANSFORMATION TO THE SCHWARZSCHILD FORM AND SINGULARITIES OF THE FIELD

If we make the transformation

$$(1+m/2r_1)^4 r_1^2 = \xi^4,$$

$$dt_1 = \left(-\frac{8r\xi\xi'}{|\dot{q}|(\xi^2-}\right)\ldots\frac{\xi^6|\dot{q}|}{2r(\xi^2-2m)} \tag{21}$$

where $m = -2K$, the solution (20) goes over to the Schwarzschild form (1) in r_1 and t_1. (It is easy to verify that dt_1 is a perfect differential.)† It may be noted that the Schwarzschild singularity $r_1 = m/2$ corresponds to $\xi^2 = 2m$, and the transformation becomes singular there and is no longer real for $\xi^2 < 2m$.

We next investigate the singularities of our field; the region $0 \leqslant r \leqslant r_1$ is obviously free from singularities except when e^q vanishes (infinite concentration). In the exterior field, the continuity of μ and μ' insures that at $r = r_1$, the left-hand side of (14) is positive and that ξ is an increasing function[11] of x. It is not difficult now to see that ξ goes on increasing and becomes arbitrarily large at a finite value of r, giving rise to a singularity of e^{μ}. However, if one calculates the proper distance of the singularity from the origin defined by $\int_0^r e^{\mu/2}dr$ or the proper volume enclosed within the singular sphere, both these quantities are found to be infinite. Further, considering the transformation (21), this singularity ($\xi \to \infty$) corresponds to the infinite sphere in the Schwarzschild coordinate system. Thus, although occurring at a finite value of r, the singularity does not appear to be anywhere in the finite region.

[9] M Wyman, Phys. Rev. 70, 74 (1946).

[10] A time scale has already been chosen in the region $r \leq r_1$ by taking $e^{\nu} = 1$, because the general solution is $e^{\nu} = f(t)$. Hence, to make the time scales on either side of the boundary identical a suitable transformation of the time variable in the region $r \geq r_1$ may be necessary. This does not, however, affect Eq (11).

† See H. P. Robertson, Trans. Am. Math. Soc. 29, 481 (1927), especially p. 492. The author is indebted to the referee of the Physical Review for this reference.

[11] Actually the condition of continuity ensures the continuity of $(1/\xi)(\partial\xi/\partial r)$, this requires that ξ and $(\partial\xi/\partial r)$ are of the same sign, but their individual signs are undetermined. A negative value of ξ leads to the same results physically and so for simplicity we restrict our considerations to the positive sign of ξ

In the next section we shall see that such a singularity occurs in the nonstatic form of even the Euclidean field.

The form (20) for e^{ν} shows that e^{ν} vanishes if $\dot\xi$ vanishes. We next investigate whether any singularity of this type is present From (14), we get

$$\xi_x = \frac{(\xi/2+3\dot q^2\xi^6/8)\xi+q\dot q\xi^6/8}{2\xi_x}$$

and

$$\frac{\dot\xi}{\xi_x}=\frac{q\dot q}{16}\int_{\xi_1}^{\xi}\frac{\xi^6 d\xi}{\xi_x^3}+\dot q\left(\frac{\xi}{4\xi_x}\right)_1 \tag{22}$$

In the above equations, the subscript x indicates differentiation with respect to x, and the subscript 1 indicates the value of the quantity at the boundary $r=r_1$ Equation (22) gives at any particular time the behavior of $\dot\xi$ (or rather of $\dot\xi/\xi_x$) as we go away from r_1.

Let us consider the asymptotic value of $\dot\xi/\xi_x$ as $\xi\to\infty$ We have, from Eq (22),

$$\left(\frac{\dot\xi}{\xi_x}\right)_{\xi\to\infty}=-\frac{q\dot q}{2q^2}\int_{\xi_1}^{\infty}d\left(\frac{\xi}{\xi_x}\right)-\frac{q\dot q m}{4q^2}\int_{\xi_1}^{\infty}\frac{d\xi}{\xi_x^3}+\dot q\left(\frac{\xi}{4\xi_x}\right)_1.$$

Since, from (14), (ξ/ξ_x) vanishes as $\xi\to\infty$, we get, after eliminating q with the help of Eqs. (10) and (16),

$$\left(\frac{\dot\xi}{\xi_x}\right)_{\xi\to\infty}=\frac{m}{\dot q}\left\{-\frac{1}{\xi_1^6(\xi_x)_1}+\frac{1}{4}\left(\frac{q^2}{2}+\frac{2m}{\xi_1^6}\right)\int_{\xi_1}^{\infty}\frac{d\xi}{\xi_x^3}\right\}. \tag{22'}$$

We note that

$$\left(\frac{\xi_x}{\xi}\right)_1=\frac{1}{2}\frac{(1-zr_1^2/4R^2)}{(1+zr_1^2/4R^2)}\equiv\alpha\quad\text{(say)}, \tag{23}$$

where α, as defined, is evidently a constant.

It is easy to see, from Eq. (9), that if we trace the history of the contracting cluster backwards in time, then for $z=-1$ or $z=0$, we can go to arbitrary large values of ξ_1 For $z=+1$, there is a maximum of ξ_1 determined by the zero of q. We get for this maximum value of ξ_1, from Eqs. (9), (16), and (23),

$$(\xi_1^2)_{\max}=2m/(1-4\alpha^2)\geqslant 2m$$

Hence, in all cases, the solution (7) allows a state at which $\xi_1^2=2m$ At this state we have

$$\int_{\xi_1}^{\alpha}\frac{d\xi}{\xi_x^3}<\int_{\xi_1}^{\alpha}\frac{64d\xi}{|q|^3\xi^9}=\frac{8}{|q|^3\xi_1^8}. \tag{23a}$$

Also, from Eqs. (14) and (23), when $\xi_1^2=2m$,

$$(\xi_x)_1=|q|\xi_1^3/4,\quad 2m/\xi_1^6=q^2/16\alpha^2. \tag{23b}$$

Substituting from (a) and (b) we find that the quantity within the brackets in Eq (22′) is definitely negative for $\alpha^2\geqslant 1/24$. Thus for $\alpha^2\geqslant 1/24$, and $\xi_1^2=2m$, $(\dot\xi)_{\xi\to\infty}$ is of the opposite sign of $\dot q$, whereas for $\xi=\xi_1$, $\dot\xi$ is of the same sign as $\dot q$. Hence for $\alpha^2\geqslant 1/24$, and $\xi_1^2=2m$. $\dot\xi$ must have a zero for some value of $\xi>\xi_1$

We can next generalize this result for arbitrary values of α and ξ_1. For we have, from Eqs. (22) and (23),

$$\frac{\ddot q}{16}\int_{\xi_1}^{\xi_s}\frac{\xi^6 d\xi}{\xi_x^3}=-\frac{1}{4\alpha},$$

where ξ_s is the value of ξ on the spherical surface at which $\dot\xi$ vanishes. From this we find that $d\xi_s/dt$ is always finite except for $e^{q}=0$ (when the field in the region $0\leqslant r\leqslant r_1$ becomes singular) and hence, since for $\alpha^2\geqslant 1/24$, and $\xi_1^2=2m$, we have a finite value of ξ_s, ξ_s will have a finite value at all stages provided $\alpha^2\geqslant 1/24$ Similarly, considering ξ_s as a function of the parameter α, we find that $\dot\xi$ will have a zero for all values of α as well.

It is thus clear that our solution, given by (20) has a singularity corresponding to the vanishing of g_{44} However, it cannot be identified with the Schwarzschild singularity for it differs in two important respects. Firstly, this singularity does not in general occur at $\xi^2=2m$ and in fact, as a little consideration shows, it does not even occur at any fixed value of ξ. Secondly, while our singularity occurs for all concentrations, the Schwarzschild singularity appears only when the concentration exceeds a certain critical value In the coordinate system which we have used, our singularity apparently is a surface separating a contracting space ($\dot\xi$ negative) in the neighborhood of the contracting cluster from an outside expanding space ($\dot\xi$ positive). However, when the solution is transformed to the static form, the singularity disappears. This suggests that the singularity has no physical reality.

Thus, in our solution there is no singularity at $\xi^2=2m$ (the Schwarzschild singularity). Further, our use of the co-moving coordinate system insures the vanishing of the coordinate velocity of the particles whose world lines are thus parallel to the time axis Hence, no question of exceeding the velocity of light arises and, unlike the situation in Einstein's cluster, there appears to be no theoretical limit to the degree of concentration.

5. CASE OF ABSOLUTELY EMPTY SPACE

In case the space is absolutely empty, K in Eq. (14) is zero and Eq. (14) on integration gives, for ξ vanishing at the origin,

$$\xi^2=2ar/(a^2-\dot q^2r^2/4),$$

where a and $\dot q$ are two arbitrary functions of time. The solution now becomes

$$e^{\mu}=4a^2/(a^2-\dot q^2r^2/4)^2,\quad e^{\nu}=\dot\mu^2/\dot q^2.$$

If one makes the transformation,

$$r'=2ar/(a^2-\dot q^2r^2/4),$$

$$dt'=-\frac{\ }{2}|\dot q|r\,dr+e\,\frac{|\dot\mu|}{2|\dot q|a}(a^2+\dot q^2r^2/4)dt,$$

the line element goes over to the pseudo-Euclidean form

$$ds^2 = -(dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\varphi^2) + dt'^2.$$

It is interesting to note that in this transformed form of even the pseudo-Euclidean field, one has a singularity at $r = 2|a|/|\dot{q}|$ where e^{μ} becomes infinite. This confirms our conclusion of the preceding section that this singularity has no physical significance. It appears only because the infinite space is mapped out in a finite coordinate region.

Singularities, of the type $g_{44} = 0$, may or may not be introduced into this solution as a and $\dot{q}$ are perfectly arbitrary functions of time

6. CONSERVATION OF GRAVITATIONAL ENERGY OF AN ISOLATED SYSTEM

In this section, we are giving up the idea of a cluster of particles and consider instead a general spherically symmetric distribution of fluid material. We have seen that (20), with ξ defined by (14), is a solution of the field equations for empty space if K be a constant Thus, any constant value of K gives rise to a solution. Further, from the transformation to the Schwarzschild form it is seen that the constancy of K means the constancy of the gravitational mass Hence, for an isolated system in empty space, the gravitational energy must be conserved. It will now be shown that in the case of spherical symmetry this condition is equivalent to the requirement of the vanishing of the pressure at the boundary. For this purpose, it is more convenient to use the spherically symmetric line element in the form

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + e^{\nu}dt^2. \qquad (24)$$

The gravitational equations are[12]

$$8\pi T_1^1 = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \qquad (25)$$

$$8\pi T_4^4 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \qquad (26)$$

$$8\pi T_{14} = \dot{\lambda}/r, \qquad (27)$$

$$-8\pi T_2^2 = -8\pi T_3^3 = e^{-\lambda}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\lambda' - \nu'}{2r}\right) - e^{-\nu}\left(\frac{\ddot{\lambda}}{2} - \frac{\dot{\lambda}\dot{\nu}}{4} + \frac{\dot{\lambda}^2}{4}\right). \qquad (28)$$

Equation (26), on integration gives

$$e^{-\lambda} = 1 - 2m/r, \qquad (29)$$

where

$$m = 4\pi\int_0^r T_4^4 r^2 dr. \qquad (30)$$

Now, the condition to fit with the external space shows that $m(r_1)$ is the gravitational mass of the system, $r_1(t)$ being the coordinate of the boundary of the sphere The rate of change of gravitational energy of the system is given by

$$\frac{dm}{dt} = m_{r_1}'\frac{dr_1}{dt} + \dot{m}_{r_1},$$

or, in view of (30), (29), and (27),

$$\frac{dm}{dt} = 4\pi r_1^2\left(T_4^4\frac{dr_1}{dt} - T_4^1\right)$$

However, if p be the pressure and ρ the density of the fluid,[13]

$$T_4^4 = \rho e^{\nu}(dt/ds)^2 + pe^{\nu}(dt/ds)^2 - p,$$
$$T_4^1 = \rho e^{\nu}(dt/ds)(dr/ds) + p(dt/ds)(dr/ds)e^{\nu},$$

so that the above equation gives

$$dm/dt = -4\pi r_1^2 p_1 dr_1/dt.$$

This is analogous to the classical relation that the rate of change of energy of an isolated system is equal to the rate of work done by the system. Thus, for a nonstatic field $[\dot{\lambda} \neq 0,\ (dr/dt) \neq 0]$,‡ the condition of conservation of the gravitational mass is equivalent to the requirement of the vanishing of pressure at the boundary.

7. CONCLUDING REMARKS

It is interesting to note why our cluster can go on contracting indefinitely, while there exists an upper bound to the concentration in the case of Einstein's cluster. The null sphere $[r = m(2+\sqrt{3})/2]$ lies beyond the Schwarzschild singularity $(r = m/2)$, so that the Einstein particles, moving in circles, are constrained to lie beyond this singularity. In our case, however, for a radially moving particle, it is known[14] that it can cross the Schwarzschild singularity in finite proper time and without attaining the velocity of light.

As is clear, there is a singularity at a finite time, the whole region $0 \leqslant r \leqslant r_1$ collapsing to zero volume. What happens after that, our equations cannot say.[15] It appears, indeed, that while we can trace the history of the birth of a particle, we cannot tell what happens when the particle is actually born This perhaps can be attributed to the fact, as remarked by Einstein,[16] that the general theory of relativity would break down under such stringent conditions

[12] Reference 6, p. 251.

[13] P. G. Bergmann, *Introduction to The Theory of Relativity* (Prentice Hall, Inc., New York, 1947), p. 129.

‡ In case $\dot{\lambda} = 0$ at the boundary, the conditions at the boundary are the same as in the static case and one gets the continuity of pressure in the usual manner.

[14] J. L. Synge, Proc. Roy. Irish Acad **A53**, 84 (1950), especially p. 107.

[15] Reference 6, p. 438.

[16] Reference 8, p. 118, footnote